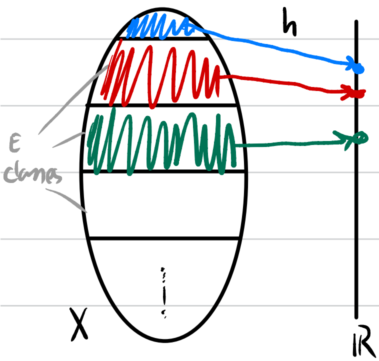
Descriptive Set Theory
Lecture 26

Classification problems.

Let X be a collection of mathematical objects (say, Riemann surfaces, measure-pres. automorphisms, C^*-algebras, ...) and let E be an equivalence relation on X (e.g. isomorphism). A classification problem is to understand the objects in X up to the equiv. rel. E. Ideally, we would like to have a "reasonable" assignment



$h : X \to \mathbb{R}$ (or some other nice space, e.g. $2^{\mathbb{N}}$) such that $\forall x_1, x_2 \in X$

$$x_1 \, E \, x_2 \iff h(x_1) = h(x_2).$$

Objects arising in analytic subjects, such as analysis, differential geometry, harmonic analysis, dynamics (measurable, topological, smooth), operator algebras (C^*, von Neumann), and functional analysis, can often be encoded into a Polish space, i.e. X is Polish. For example, we already saw that $K([0,1]^{\mathbb{N}})$ and $F(\mathbb{R}^{\mathbb{N}})$ can be thought of as the Polish spaces of compact separable spaces and of Polish spaces, resp.

Moreover, the eq. rel. E that we care about are most often analytic (as subsets of X^2) because they are defined using an existential quantifier over a Polish followed by something Borel. As for the assignment h, one can always use axiom of choice to pick a point from each class and get a desired function to $\mathbb{R}$ by cardinality considerations. Demanding h to be continuous is too much because then building h may be obstructed by the top. on X which has nothing to do with the complexity of the classification problem, i.e. the complexity of E. So we demand h to be Borel.

Def. An eq. rel. E on a stand. Borel space X is called *concretely classifiable* (or *smooth*) if $\exists$ Borel reduction h of E to equality $=_{\mathbb{R}}$ on $\mathbb{R}$ (equiv. any other unctbl st. Borel space), i.e. $\exists\, h: X \to \mathbb{R}$ s.t. $\forall x_1, x_2 \in X$,

$$x_1 E x_2 \iff h(x_1) = h(x_2).$$

In other words, h descends to an embedding $X/_E \hookrightarrow \mathbb{R}$.

More generally, solving a classification problem E on X means to understand the "Borel cardinality" of X/E, i.e. we say $|X/E| \leq_B |Y/F|$, where B stands for Borel, if $\exists$ Borel map $h : X \to Y$ that descends to an injection $X/E \hookrightarrow Y/F$. In particular, for E on X, X/E may larger Borel cardinality than X.

Def. We say that an eq. rel. E on a st. Borel X is *Borel reducible* to an eq. rel. F on a st. Bor. Y, if $\exists$ Borel reduction h of E to F, i.e. $\exists\, h : X \to Y$ s.t. $\forall x_1, x_2 \in X$,

$$x_1 E x_2 \iff h(x_1) F h(x_2).$$

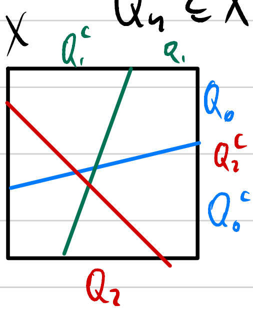
In other words, h descends to an embedding $X/E \hookrightarrow Y/F$. We denote this by $E \leq_B F$.

The study of Borel reducibility of analytic eq. rel. has become its own subject that provides classification and anti-classification results to the aforementioned areas of math. This happened in the last 30 years, initiated by Alexander Kechris, A. Louveau, Greg Hjorth, and others.

Note that if E on X is smooth then E is Borel (as a subset of X^2): $E = \tilde{h}^{-1}(\Delta_{\mathbb{R}})$, where $\Delta_{\mathbb{R}}$ = diagonal and $\tilde{h}(x_1, x_2) := (h(x_1), h(x_2))$, so $\tilde{h}$ is Borel.
This "means" that there is a ctbl algorithm s.t. given $x_1, x_2 \in X$, determines whether $x_1 E x_2$. In fact, we can take a sequence of yes/no Borel questions s.t. $x_1 E x_2$ iff x_1 and x_2's answers to these questions are the same:

Prop. E on X is smooth $\iff$ $\exists$ Borel sets (questions) $Q_n \subseteq X$ s.t. $\forall x_1, x_2 \in X$,

$$x_1 E x_2 \iff \forall n \, (x_1 \in Q_n \iff x_2 \in Q_n).$$



Proof. $\Leftarrow$. We define a Borel reduction of E to $=_{2^{\mathbb{N}}}$ $h_Q : x \mapsto (a_n^x)_n$, where

$$a_n := \begin{cases} 0 & \text{if } x \notin Q_n \\ 1 & \text{if } x \in Q_n \end{cases}.$$

Indeed, then $(a_n^{x_1})_n = (a_n^{x_2})_n \iff x_1 E x_2$.

$\Rightarrow$. Suppose $\exists$ Borel reduction $h : X \to 2^{\mathbb{N}}$ of E to $=_{2^{\mathbb{N}}}$.
Define $Q_n := h^{-1}([* * * \dots * \underset{\substack{\uparrow \\ n}}{1} * * \dots])$. $\square$

Examples of smooth eq. rel.

(a) Isomorphism of fin. gen. abel. gps. Firstly, we encode all ctbl gps with the underlying set $\mathbb{N}$ into a Polish space as follows: a group Γ on $\mathbb{N}$ is a structure $(\mathbb{N}, \cdot)$ where $\cdot$ is a binary op, satisfying some axioms. Replacing $\cdot$ by its graph, i.e. a ternary relation $R_\cdot \subseteq \mathbb{N}^3$, we get a relational structure $(\mathbb{N}, R_\cdot)$, so Γ can be decoded from $R_\cdot$. $R_\cdot \in 2^{(\mathbb{N}^3)}$ and those $R_\cdot \in 2^{(\mathbb{N}^3)}$ that satisfy the group axioms form a closed set. Thus, $Gp(\mathbb{N})$ is a compact Polish space. The space FCAG of fin. gen. abel. gp form a Σ^0_3 subset of $Gp(\mathbb{N})$, hence it's a st. Borel space. We know from algebra that every fin. gen. ab. gp Γ is isom. to a gp of the form $\mathbb{Z}^n \times$ (fin. ab. gp). Turns out that the map $\Gamma \mapsto (n, \text{fin ab gp})$ is Borel, witnessing the smoothness of the isom. rel. on FCAG.

(b) Let $M_n(\mathbb{C}) \cong \mathbb{C}^{(n^2)}$ be the (Polish) space of all $n \times n$ complex matrices. Let $\sim$ denote the similarity of matrices, i.e.

matrices $A \sim B :\Leftrightarrow A, B$ are conjugate
$\Leftrightarrow \underbrace{\exists Q \in GL_n(\mathbb{C}) \; \underbrace{QAQ^{-1} = B}_{\text{closed}}}_{\text{analytic}}$.

By def., $\sim$ is an analytic eq. rel.

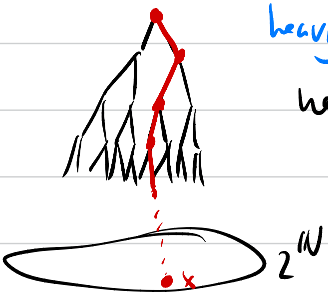
Letting $J(A)$ denote the Jordan canonical form of A, we know from lin. alg. that $A \sim B \Leftrightarrow J(A) = J(B)$. Again, one can show that the map $A \mapsto J(A)$ is Borel, witnessing the smoothness (hence also Borelness) of $\sim$.

Prop. Let E be an eq. rel. on a Polish X. If E is generically erg. (i.e. E-inv. Borel sets are meager or comeager) and each E-class is meager, then E is nonsmooth. Similarly, if E is μ-ergodic and each E-class is μ-null, for some Borel measure μ on X, then E is nonsmooth.

Proof. To prove both at once, call meager/null sets small. Suppose E is smooth, so $\exists$ Borel $h: X \to 2^{\mathbb{N}}$ s.t. $\forall x_1, x_2 \in X$,

$$x_1 E x_2 \Longleftrightarrow h(x_1) = h(x_2),$$

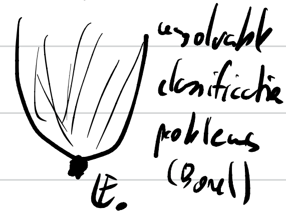
i.e. h is constant on each E-class, in particular, $h^{-1}(B)$ is E-invariant Borel, for each $B \subseteq 2^{\mathbb{N}}$. Thus, $h^{-1}(B)$ is small or cosmall. Call an $s \in 2^{<\mathbb{N}}$ heavy if $h^{-1}([s])$ is cosmall. If s is heavy then $\exists$ (unique) $i \in \{0,1\}$ s.t. $s^\frown i$ is heavy. Start with $\emptyset$, which is heavy and chase it down to $x \in 2^{\mathbb{N}}$, following the heavy child. But $\{x\} = \bigcap_n [x|_n]$ so $h^{-1}(x) = \bigcap_n h^{-1}([x|_n])$ is cosmall, but $h^{-1}(x)$ is at most one E-class, which is small, a contradiction. $\square$



Examples. $E_{\mathbb{Q}}$ ($\mathbb{Q} \curvearrowright \mathbb{R}$ by translation), irrational rotation are nonsmooth. So is $\mathbb{E}_0$ on $2^{\mathbb{N}}$ defined by $x \,\mathbb{E}_0\, y \iff \forall^\infty_n \; x(n) = y(n)$ (eventual equality).

Note that if $\mathbb{E}_0 \le_B E$, then E is also nonsmooth (otherwise, composition would witness smoothness of $\mathbb{E}_0$). Turns out, this is the only obstruction to smoothness for Borel eq. rel.

$\mathbb{E}_0$-dichotomy (Kechris-Harrington-Louveau). For any Borel eq. rel. E on a st. Borel X



either E is smooth, i.e. $E \leq_B =_{2^{\mathbb{N}}}$.

or $\mathbb{E}_0 \sqsubseteq_B E$.

This shows that $\mathbb{E}_0$ is the minimum element among non smooth Borel eq. rel. This dich. generalizes earlier thms of Glimm and Effros, so it is referred as the generalized Glimm-Effros dichotomy.
The original proof of this dichotomy uses effective descriptive theory, i.e. a finer top. on X that comes from Turing machines.